Higher Algebra

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The goal is to understand why it is reasonable to consider the (infinity) category of spectra as a generalisation of various classical algebraic categories (rings, abelian groups). We will start from the most general setting first.

1 Deriving an Abelian Category

In traditional homological algebra one might consider an abelian category $\mathcal{A}$ with enough projectives. To such a category we can associate the category of chain complexes which might be denoted $\mathrm{Ch}\mathcal{A}$. This is where most of homological algebra takes place, now two chain complexes are called quasi-isomorphic iff there is a map beetween them that produces an isomorphism on co/homology. Now quasi-isomorphisms of chain complexes, along with epimorphisms provide the category with a model structure (weak equivalences and fibrations respectively). The bifibrant objects in this model structure are the chains where all the chain objects are projective objects of $\mathcal{A}$. The homotopy category of this model structure can be formed and is usually called $\mathcal{D}(\mathcal{A})$ the derived category of $\mathcal{A}$, and we can always consider this as consisting of chains of projective objects. This is all done because in homological algebra we are only interested in our chain complexes up to quasi-isomorphism, say for topological spaces, we care about cohomology not the singular chain complex.

The classical story then goes on like this: the derived category has a triangulated structure and triangulated categories can further have a t-structure [GM96, IV.4.1]. For the derived category it is easy to state the t-structure directly, it is two full subcategories satisfying some properties, in the derived setting we merely take the two categories of chain complexes with non-trivial homology in positive degrees and negative degrees as the two categories. Their intersection is then clearly the chain complexes with non-trivial homology in degree zero. This intersection is called the *heart* or *core* of the t-structure and in general we have

Lemma (III.5.2, [GM96])**.** *There is an equivalence of categories* $\mathcal{A} \to \mathcal{D}(\mathcal{A})^{\geq 0} \cap \mathcal{D}(\mathcal{A})^{\leq 0}$.

Thus in the one categorical setting the derived category is a strict enlargment of the original abelian category. Now we will see that this strategy also holds in the infinity category world.

Definition (Def 1.3.2.7 [Lur17])**.** *Lurie defines the derived infinity category of of an abelian (one) category. The construction is similar to that of the coherent nerve and is given in [Lur17, Construction 1.3.1.6]. The objects and morphisms are the same as those of the traditional derived category. The two morphisms are the chain homotopies etc.*

In the context of infinity categories the structure of a triangulation is replaced by the *property* of being stable [Lur17, Def 1.1.1.9, Cor 1.4.2.27] (note that the one category of spectra is also triangulated). The two definitions here are bridging the algebraic (exact triangles) and the topological (suspension isomorphism). A stable infinity category has a triangulated homotopy category [Lur17, Thm 1.1.2.14], and a t-structure on this infinity category is then a t-structure on the homotopy category. This is both useful and potential justification for the idea that stable categories gernalise triangulated categories. It allows us to define the heart of a stable infinity category as the full subcategory on those objects that are in the heart of its homotopy category.

Lemma (Prop 1.3.2.10, [Lur17])**.** *The derived category is stable. Its heart is canonically equivalent to the nerve of $\mathcal{A}$.*

So in one categories the derived category is a strict generalisation of some abelian category. Infinity categorically then we also have a way to send an abelian category to a stable infinity category that is moreover reversible some sense.

Remark. Even though $\mathcal{A}$ is equivalent to the heart of its derived category, and the derived category consists of only projective module chains, $\mathcal{A}$ is not equivalent to some subcategory of projective modules. $\mathcal{A}$ is maped to chains with only projective modules but there is homology in the zeroeth degree, thus $\mathcal{A}$ is equivalent to some category of *quotients* of projective modules.

Remark. Note that the process of taking the derived category

$$\text{Abelian Categories} \underset{\heartsuit}{\overset{\mathcal{D}}{\rightleftarrows}} \text{Triangulated categories}$$

is an inverse from LHS to RHS and back to LHS, but even on the level of one categories the RHS is bigger. This post gives an example of triangulated categories that are not the derived categories of their hearts.

2 Stability and Spectra

So we have seen that stable infinity categories generalise abelian categories. We will return to that construction, here however we want to make a few definitions for what will be our second enlargment, that is algebraic structures in Spectra.

Definition. *Ring spectra, modules over a ring spectrum.*

A spectrum is called connective if its only non-zero homotopy group is in degree zero.

$\mathbb{E}_{k+1}$-ring R ($\mathbb{E}_{k+1}$ algebra in Spectra)

Define $\mathrm{Alg}_R^{(k)}$ and $\mathrm{LMod}_R^{\mathrm{perf}}$

π_0 of a conncective ring spectrum is an ordinary ring.

Now we need to see how spectra fit in.Now spectra historically have been the model of stability, but can we make the relation precise

Remark. WHat is the spectrum (Spec) of a ring spectrum

3 Rings

Finally we will connect the two constructions we have made. First recall that for (small) abelian categories Freyd-Mitchel implies that it is equivalent to a full subcategory of a module category of some ring. Thus we will restrict our attention now to module categories.

Lemma (Prop 7.1.3.18[Lur17])**.** *Given a connective $\mathbb{E}_{k+1}$-ring R then there is an equilvalence*

$$\mathrm{Alg}_R^{(k)} \simeq N\mathrm{Alg}_{\pi_0 R}$$

A similar theorem applies to the derived category, by page 2 of these notes, we have that for a discrete A_∞ ring spectrum R there is an equivalence of infinity categories

$$\mathrm{LMod}_R^{\mathrm{perf}} \simeq \mathcal{D}^{\mathrm{perf}}(\pi_0(R))$$

Thus if we have a ring spectrum that is discrete we can recover the derived category of the ordinary ring or the category of modules over that ring (that is the abelian category) that is its π_0. On the other hand we should have a way to *produce* a discrete ring spectrum from an ordinary ring, and these should be inverses.

Definition (Ex 1.3.4, Ex 5.3.2, [BR20])**.** *There are two constructions. The easiest to state is the sequential spectrum, in which the spaces are just the Eilenberg-Maclane spaces for the ring.*

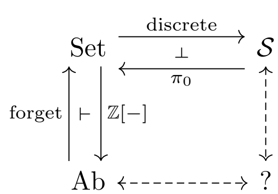
By [BR20, Ex 2.2.4] the Eilenberg-Maclane spectrum is discrete with only non-zero homotopy group in degree zero, given by the group. When the group is the additive group of a ring then the Eilenberg-Maclane spectrum is a ring spectrum by [BR20, Ex 5.3.2].

Thus for an abelian category, or a ring, we can either take the derived category or we can create an Eilenberg-Maclane spectrum from it. We have seen that the derived category construction allows us to recover the ring. Now we have seen that taking the EM spectrum and its category of modules (as spectra) allows us to recover *the derived category* and hence recover the original ring or abelian category. Thus ring spectra and modules over ring spectra are a suitable generalisation or rings and modules!

Remark. Using this it is possible to see how a ring is generalised by a ring spectrum. Since Rings $\cong$ $\mathbb{Z}-$alg fully faithfully embeds in the category of $H\mathbb{Z}$-algebras we can see that two rings are isomorphic iff their corresponding $H\mathbb{Z}$ algebras are isomorphic as $H\mathbb{Z}$ algebras.

4 Naive Homotopical Algebra

This is the story I was told by Lior, he insists that one should think of spectra *as the homotopy version of commutative groups*. Math starts out in sets, that is with no structure. There are two directions that it generalises, the topological (homotopical) and the algebraic. These generalisations can be fit into a conceptual and also functorial diagram why do we take π_0 and not forget the topology?



Spectra should be the thing that does *both*. An obvious candidate would be the infinity category $\mathcal{D}(\mathrm{Ab})$, the derived category of abelian groups. The derived category is an example of so called 'animation',

$$\mathcal{D}_{\geq 0}(\mathrm{Ab}) \simeq \mathrm{Ani}(\mathrm{Ab}) = \mathrm{Fun}^{\times}(\mathrm{Ab}^{\mathrm{op}}_{\text{free, f.g.}}, \mathcal{S}).$$

This is (basically) just the free (sifted) cocompletion of $\mathrm{Ab}^{\mathrm{op}}_{\text{free, f.g.}}$, and comes from the one categorical motivation that for instance $\mathrm{Ab} \simeq \mathrm{Fun}^{\times}(\mathrm{Ab}^{\mathrm{op}}_{\text{free, f.g.}}, \mathrm{Set})$. This is not the most homotopical thing that we could have done however, as we have just replaced the codomain of our functors with spaces, what about the domain?

To avoid technical dificulty we will talk now about the category CMon of commutative monoids. This also has a free finitely generated subcategory that can be conflated with $\mathbb{N}$, the number of generators. The hom sets can be seen to be matricies $\mathrm{Hom}(n, m) = \mathrm{Mat}_{n\times m}(\mathbb{N})$. Whenever a homotopy theorist sees $\mathbb{N}$ their eyes light up and they say "$\mathrm{Fin}^{\mathrm{iso}}$ (category of finite sets with only isomorphisms as morphisms) is the free commutative monoid in spaces" and $\pi_0\mathrm{Fin}^{\mathrm{iso}} \cong \mathbb{N}$. Thus $\mathbb{N}$ is a decategorifies $\mathrm{Fin}^{\mathrm{iso}}$. Now we know what we need to do, we need to replace $\mathbb{N}$ by $\mathrm{Fin}^{\mathrm{iso}}$. We make the hom sets then $\mathrm{Mat}_{n\times m}(\mathrm{Fin}^{\mathrm{iso}})$, where now we add and multiply finite sets using disjoint union and cartesian product. These additions and products will only associate up to isomorphism and therefore the category $\mathrm{Mat}(\mathrm{Fin}^{\mathrm{iso}})$, with objects $\mathbb{N}$ and morphisms as described, is naturally a $(2, 1)$-category. Thus we can generalise the animation process to define for instance

$$\mathrm{CMon}(\mathcal{S}) \quad := \quad \mathrm{Fun}^{\times}(\mathrm{Mat}(\mathrm{Fin}^{\mathrm{iso}}), \mathcal{S})$$

where the functors obviously are as infinity categories. To get back to abelian groups we just take the group like objects in this category, and this gives us

$$\mathrm{CGrp}(\mathcal{S}) \quad := \quad \mathrm{CMon}(\mathcal{S})^{\mathrm{gp}}$$

which is the model of $\mathrm{Sp}^{\mathrm{cn}}$ connective spectra. This gives the naive idea "spectra are commutative homotopical groups".

This is the sort of algebraic to homotopical angle to define spectra, from the stabilisation perspective, that is stabilising the category of spaces, we know however that this category of connective spectra sits inside a much larger category. In particular the sphere spectrum is not connective. From the topological perspective, we want to start with a space and go down in our diagram which is sort of increase the amount of algebraic structire it has, that is we want to see the homotopical to algebraic perspective. Topologically algebraic objects come from *loop spaces*. In finite loop spaces are the most commutative objects, with their homotopy commutative homotopy coherent operations satisfying all higher commutativity relations. Classically an infinity loop space can be seen to be equivalent to a connective sequential spectrum. This definition generalises straight away to the infinity categorical perspective as we can set

$$\mathrm{Sp} \quad := \quad \lim_{\leftarrow} \big(\cdots \xrightarrow{\Omega} \mathcal{S} \xrightarrow{\Omega} \mathcal{S} \xrightarrow{\Omega} \mathcal{S} \big)$$

This is the full category of spectra, as we have that

$$\mathrm{Sp}^{\mathrm{cn}} \quad := \quad \lim_{\leftarrow} \big(\cdots \xrightarrow{\Omega} \mathcal{S}^{1-\mathrm{cn}} \xrightarrow{\Omega} \mathcal{S}^{0-\mathrm{cn}} \xrightarrow{\Omega} \mathcal{S} \big).$$

Remark. (Terminology) It seems that what infinity category people call a spectrum is what [BR20] might call an Ω-spectrum. Recall that Σ^{∞} sends the category of infinite loop spaces to connected spectra. The limit we have defined unpacked defines an Ω-spectrum, which can be neither connected nor connective in general.

References

[BR20] David Barnes and Constanze Roitzheim. *Foundations of Stable Homotopy Theory.* Cambridge University Press, 1 edition, March 2020.

[GM96] Sergei I. Gelfand and Yuri I. Manin. *Methods of Homological Algebra.* Springer Berlin Heidelberg, Berlin, Heidelberg, 1996.

[Lur17] Lurie. *Higher Algebra.* 2017.